On the greatest zero of an orthogonal polynomial. II

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1. Introduction

In the present paper we are continuing our investigations initiated in [2]. (See also [3] and P. G. NÉVAI [5].)

As in the papers mentioned, we denote by $p_n(W; x)$ the nth degree orthogonal polynomial with respect to the weight $W(x)$ $(-\infty<x<\infty)$ and by $X_n(W)$ we denote the greatest zero of $p_n(W; x)$. Through all our present paper we assume that $W(x)$ is even.[1])

The most typical result of this paper is

Theorem 1. *Let*

$$W_Q(x) = \exp\{-2Q(x)\} \tag{1}$$

where $Q(x)$ is an even differentiable function, increasing for $x>0$, for which $x^{\varrho}Q'(x)$ is increasing for some $\varrho<1$ then we have

$$c_1 q_n \leqq X_n(W_Q) \leqq c_2 q_n \tag{2}$$

where c_1, c_2 do not depend on n, and q_s $(s>0)$ is determined by the equation

$$q_s Q'(q_s) = s. \tag{3}$$

Let us observe that as a consequence of our assumption $xQ'(x)$ is also increasing for $x>0$ so that the sequence $\{q_s\}$ is well defined.

We obtain Theorem 1 as a consequence of far more general but slightly technical estimates (see Theorem 2 and Theorem 3).

Theorem 1 is applicable for the case $Q_\alpha(x)=\frac{1}{2}|x|^\alpha$ $(\alpha>0)$ and we obtain $X_n(W_{Q_\alpha})\sim n^{1/\alpha}$. This was proved earlier for positive even integer values of α by the author in [2] and for general $\alpha>0$ by G. P. NÉVAI [5].

[1]) Estimates for the zero with greatest modulus of $p_n(W; x)$ are obtained in the general case by combining the result of this paper by Lemma 7 of our preceding paper [3].

We conjectured in our paper [3] that an inequality similar to (2) might hold under the less restrictive condition that $xQ'(x)$ is increasing for $x>0$. [2]) This problem remains unsettled.

2. Preliminary estimates

In all our paper we assume that $W(x)$ is an even continuous positive function on the whole real line. It follows that

$$G_\xi(W) = \exp\left\{\frac{1}{\pi}\int_0^\pi \log\left[W(\xi\cos\theta)\right] d\theta\right\} \tag{4}$$

is well defined and finite for every $\xi \geqq 0$.

We need also the trunkated weights

$$W_\xi(x) = \begin{cases} W(x) & (|x| \leqq \xi) \\ 0 & (|x| > \xi). \end{cases} \tag{5}$$

Lemma 1. *The leading coefficients $\gamma_\nu(W)$ resp. $\gamma_\nu(W_\xi)$ of the orthogonal polynomials $p_\nu(W; x)$ resp. $p_\nu(W_\xi; x)$ satisfy for every $\xi>0$*

$$\gamma_\nu(W) \leqq \gamma_\nu(W_\xi) \leqq \sqrt{\frac{2}{\pi}}\,\xi^{-1/2}\left(\frac{2}{\xi}\right)^\nu [G_\xi(W)]^{-1/2}. \tag{6}$$

Proof. (See also [4], part 2.) We have

$$\int_{-1}^{1} |p_\nu(W_\xi; \xi t)|^2 W(\xi t)\, dt = \xi^{-1}\int_{-\xi}^{\xi} p_\nu^2(W_\xi; x) W_\xi(x)\, dx = \xi^{-1}$$

and by setting $z = e^{i\theta}$, $t = \cos\theta = \frac{1}{2}(z+z^{-1})$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|z^\nu p_\nu\left[W_\xi; \frac{1}{2}\xi(z+z^{-1})\right]\right|^2 W(\xi\cos\theta)\,|\sin\theta|\, d\theta = \frac{1}{\pi\xi}. \tag{7}$$

Since $W(x)$ is positive and continuous we have $\log W(\xi\cos\theta) \in L$ for every

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[2]) In general q_s and q_{2s} have not the same order of increase for $s\to\infty$. As a consequence of the reasonable hypothesis for the general case is that there exist positive numbers c_3, c_4, c_5 and c_6 depending only on the choice of W_Q for which we have

$$c_3 q_{c_4 n} \leqq X_n(W_Q) \leqq c_5 q_{c_6 n}.$$

As a consequence of our additional assumption that $x^\varrho Q'(x)$ is increasing for a $\varrho<1$ we have $q_{cn}\sim q_n$ for every fixed $c>0$. (See (31).)

fixed $\xi>0$. By a theorem of G. SZEGŐ (see e.g. [6], part 10) there exists a function $D(z)\in H_2$ satisfying a.e.

(8) $$|D(e^{i\theta})|^2 = W(\xi\cos\theta)\,|\sin\theta|$$

and

(9) $$D(0) = \exp\left\{\frac{1}{2\pi}\int_0^\pi \log[W(\xi\cos\theta)\,|\sin\theta|]\,d\theta\right\} = \frac{1}{\sqrt{2}}[G_\xi(W)]^{1/2}.$$

Now

(10) $$P_\nu(z) = z^\nu p_\nu\left[W_\xi;\ \frac{1}{2}\xi(z+z^{-1})\right]$$

is a polynomial of degree 2ν for which we have

(11) $$P_\nu(0) = \left(\frac{\xi}{2}\right)^\nu \gamma_\nu(W_\xi).$$

By construction $P_\nu D\in H_2$ so that by (9) and (11) resp. by (7), (8), and (10)

$$\frac{1}{\sqrt{2}}[G_\xi(W)]^{1/2}\left(\frac{\xi}{2}\right)^\nu \gamma_\nu(W_\xi) = D(0)P_\nu(0) \leqq \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|P_\nu(e^{i\theta})D(e^{i\theta})|^2\,d\theta\right\}^{1/2} = (\pi\xi)^{-1/2}.$$

By reshuffling terms in this inequality we obtain the second part of (6). The first part of (6) follows from $W_\xi(x)\leqq W(x)$, q.e.d.

We turn to investigate the Christoffel functions (see e.g. [1])

(12) $$\lambda_n(W;x) = \left\{\sum_{\nu=0}^{n-1} p_\nu^2(W;x)\right\}^{-1}.$$

Lemma 2. *We have for every pair every (x,ξ) with $|x|>\xi>0$*

(13) $$\lambda_n^{-1}(W;x) \leqq \lambda_n^{-1}(W_\xi;x) \leqq \frac{4}{3\pi}[\xi G_\xi(W)]^{-1}\left(\frac{2|x|}{\xi}\right)^{2n-2}.$$

Proof. The first part of (13) is a consequence of $W_\xi(x)\leqq W(x)$. To prove the second part first we observe that all zeroes $x_{k\nu}=x_{k\nu}(W_\xi)$ of $p_\nu(W_\varrho;x)$ are contained in $(-\xi,\xi)$ (since $W_\xi(x)=0$ for $|x|>\xi$) and the $x_{k\nu}$'s are distributed symmetrically around the origin (since W_ξ is even). Consequently we have for every natural ν

(14) $$|p_\nu(W_\xi;x)| = \gamma_\nu(W_\xi)\,|x|^{\nu-2[\nu/2]}\prod_{x_{k\nu}>0}(x^2-x_{k\nu}^2) \leqq$$
$$\leqq \gamma_\nu(W_\xi)\,|x|^\nu \leqq \sqrt{\frac{2}{\pi}}\left(\frac{2}{\xi}\right)^\nu \xi^{-1/2}[G_\xi(W)]^{-1/2}|x|^\nu$$

the last part in consequence of (6).

By (14) we have

$$(15)\quad \lambda_n^{-1}(W;x) \leqq \lambda_n^{-1}(W_\xi;x) = \sum_{\nu=0}^{n-1} p_\nu^2(W_\xi;x) \leqq \frac{2}{\pi}[\xi G_\xi(W)]^{-1} \sum_{\nu=0}^{n-1}\left(\frac{2|x|}{\xi}\right)^{2\nu} =$$

$$= \frac{2}{\pi}[\xi G_\xi(W)]^{-1} \frac{\left(\frac{2|x|}{\xi}\right)^{2n}-1}{\left(\frac{2|x|}{\xi}\right)^{2}-1}.$$

Under the assumption $|x|>\xi$ (15) implies (13), q.e.d.

Remark. We assumed in the proof only that $W(x)$ is nonnegative and that $G_\xi(W)$ is finite. The example of Chebychev polynomials and $|\xi|=1$, $|x|\to\infty$ shows that under this less stringent assumptions the factor $2/\pi$ in (15) is best possible for every $n\geqq 2$. By a continuity argument it follows that $2/\pi$ is the best possible factor even if only continuous positive weights are admitted.

3. The fundamental estimates for $X_n(W)$

Theorem 2. *We have for every* $\xi>0$ *and every* $A\geqq 1$

$$(16)\qquad X_n(W) \leqq A\xi + \frac{4}{3\pi}\left(\frac{2}{\xi}\right)^{2n-1}[G_\xi(W)]^{-1}\int_{A\xi}^{\infty} x^{2n-1}W(x)\,dx.$$

Proof. By Chebychev's theorem

$$(17)\qquad X_n(W) = \sup_{P\in\Phi_{n-1}} \int_{-\infty}^{\infty} x[P(x)]^2 W(x)\,dx$$

where Φ_{n-1} is the set of polynomials $P(x)$ of degree $n-1$ at most for which we have

$$(18)\qquad \int_{-\infty}^{\infty}[P(x)]^2W(x)\,dx \leqq 1.$$

As a consequence of (18) we have

$$(19)\qquad [P(x)]^2 \leqq \lambda_n^{-1}(W;x) \quad (-\infty<x<\infty).$$

By (19) and (13)

$$(20)\quad \left|\left\{\int_{-\infty}^{-A\xi}+\int_{A\xi}^{\infty}\right\} x[P(x)]^2W(x)\,dx\right| \leqq 2\frac{4}{3\pi}\xi[G_\xi(W)]^{-1}\int_{A\xi}^{\infty} x\left(\frac{2x}{\xi}\right)^{2n-2}W(x)\,dx =$$

$$= \frac{4}{3\pi}\left(\frac{2}{\xi}\right)^{2n-1}[G_\xi(W)]^{-1}\int_{A\xi}^{\infty} x^{2n-1}W(x)\,dx.$$

By (18) we have

(21) $$\int_{-A\xi}^{A\xi} x[P(x)]^2 W(x)\,dx \leqq A\xi \quad (P\in\Phi_{n-1}).$$

Now (16) follows from (17), (20) and (21), q.e.d.

Theorem 3. *We have for every even nonnegative W and every $\xi>0$*

(22) $$X_n(W) \geqq \frac{1}{2}\left\{\frac{2}{\pi}\int_{-\infty}^{\infty} W(x)\,dx\right\}^{-\frac{1}{2n-2}} \{\xi^{2n-1}G_\xi(W)\}^{\frac{1}{2n-2}}.$$

Proof. By [2] we have

$$X_n(W) \geqq \frac{\gamma_{\nu-1}(W)}{\gamma_\nu(W)} \quad (\nu = 1, 2, \dots, n-1).$$

Consequently

(23) $$[X_n(W)]^{n-1} \geqq \prod_{\nu=1}^{n-1} \frac{\gamma_{\nu-1}(W)}{\gamma_\nu(W)} = \frac{\gamma_0(W)}{\gamma_{n-1}(W)} = \frac{1}{\gamma_{n-1}(W)\left\{\int_{-\infty}^{\infty} W(x)\,dx\right\}^{\frac{1}{2}}};$$

(22) is implied by (6) and (23), q.e.d.

4. Special cases of the fundamental estimates

Let us assume that $W(x)$ is even, continuous, positive, *decreasing for* $x>0$ and that for every natural ν

(24) $$\lim_{x\to\infty} x^\nu W(x) = 0.$$

For such W and for every ν there exists a smallest ξ_ν satisfying

(25) $$\xi_\nu^\nu W(\xi_\nu) = \max_{x\geqq 0} x^\nu W(x).$$

Theorem 4. *Under the stated assumptions concerning W we have*

(26) $$\frac{1}{2}\left\{\frac{\pi}{2\int_{-\infty}^{\infty} W(x)\,dx}\right\}^{\frac{1}{2n-2}} [\xi_{2n-1}^{2n-1} W(\xi_{2n-1})]^{\frac{1}{2n-2}} \leqq X_n(W) \leqq \left(2+\frac{1}{3\pi}n^{-1}\right)\xi_{4n}.$$

Proof. Since $W(x)$ is decreasing for $x>0$ we have

(27) $$G_\xi(W) \geqq W(\xi).$$

Inserting $\xi=\xi_{2n-1}$ in (22) and taking (27) in consideration we obtain the left hand side of (26). To obtain the right hand side, let us insert in (16) $A=2$ and $\xi=\xi_{4n}$, thus by (27)

$$\int_{2\xi_{4n}}^{\infty} x^{2n-1}W(x)\,dx \leqq \xi_{4n}^{4n}W(\xi_{4n})\int_{2\xi_{4n}}^{\infty} x^{-2n-1}\,dx \leqq \frac{1}{2n\,2^{2n}}\xi_{4n}^{2n}G_\xi(W). \tag{28}$$

From (28) and (16) with $A=2$, $\xi=\xi_{4n}$ we get the second half of (26), q.e.d.

Proof of Theorem 1.[3]) By (1), (3) and (25) we have $\xi_\nu=q_{\nu/2}$. Since $x^\varrho Q'(x)$ is increasing we have by assumption for every $0<s<S$

$$\frac{S}{s}=\frac{q_S Q'(q_S)}{q_s Q'(q_s)}\geqq\left(\frac{q_S}{q_s}\right)^{1-\varrho},$$

i.e.

$$q_s\leqq q_S\leqq\left(\frac{S}{s}\right)^{\frac{1}{1-\varrho}}q_s\quad(0<s<S). \tag{29}$$

Assuming $\xi_s\geqq 1$ we have by (3)

$$Q(\xi_s)\leqq Q(1)+\xi_s^\varrho Q'(\xi_s)\int_1^{\xi_s}t^{-\varrho}\,dt<Q(1)+\frac{s}{1-\varrho}$$

and this is valid also for $0<\xi_s<1$ since $Q(x)$ is increasing. Consequently

$$W(\xi_{2n-1})=e^{-2Q(\xi_{2n-1})}\geqq e^{-2Q(1)-2(1-\varrho)^{-1}(2n-1)}=W(1)e^{-2(1-\varrho)^{-1}(2n-1)}. \tag{30}$$

The estimates (2) follow from (26), (29) and (30), q.e.d.

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[3]) See the Introduction.